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Modulated waves in Taylor–Couette flow Part 1. Analysis

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We present a mathematical analysis of the transition from temporally periodic rotating waves to quasi-periodic modulated waves in rotating flows with circularly symmetric boundary conditions, applied to the flow between concentric, rotating cylinders (Taylor–Couette flow). Quasi-periodic flow (modulated wavy vortex flow) is described by two incommensurate, fundamental, temporal frequencies in an arbitrary frame, but the flow is periodic in the appropriate rotating frame. The azimuthal wavelength of the modulation may be different to that of the underlying rotating wave; hence the flow state is described by two azimuthal wavenumbers as well. One frequency and one wavenumber are determined by the wave state, but no simple physical properties have yet been associated with the parameters of the modulation. The current literature on modulated waves displays both conflicting mathematical representations and qualitatively different kinds of modulation. In this paper we use Floquet theory to derive the unique functional form for all modulated waves and show that the space–time symmetry properties follow directly. The flow can be written as a non-separable function of the two phases $(\theta - c_1 t, \theta - c_2 t)$. We show that different branches of modulated wave solutions in Taylor–Couette flow are distinguished not by symmetry but by the ranges of the numerical values of c_1, c_2, and the spectral amplitudes of the solution. The azimuthal wavenumber associated with the modulation has a unique physical definition but is not directly expressed in the spatial symmetry of the modulated flow. Because modulated waves should occur generically in systems with rotational symmetry, this analysis has application beyond Taylor–Couette flow.

1. Introduction

The flow between concentric, differentially rotating cylinders, Taylor–Couette flow, exemplifies a class of problems in fluid dynamics involving rotating flows with circularly symmetric boundary conditions. When the outer cylinder is held fixed and the inner cylinder speed is increased, the flow undergoes a series of transitions (the 'main sequence' (Golubitsky & Stewart 1986)), which are characterized by the change in the symmetry group that leaves the flow invariant. In particular, in such systems a temporally periodic rotating wave will generically bifurcate to a quasi-periodic modulated wavy flow. The symmetry group defining these quasi-periodic flows, consisting of a lattice of space–time translations which bring the flow into itself, was derived by Rand (1982) and confirmed in experiments of Gorman & Swinney (1979, 1982) on modulated wavy Taylor vortex flow (MWV). Work on this system has been motivated in a large part by interest in the transition to chaos, which has been shown in the laboratory to occur directly out of the MWV state as the inner cylinder speed is increased (Gollub & Swinney 1975; Fenstermacher, Swinney & Gollub 1979). It was believed that this transition might be related to the work of Ruelle & Takens (1971), who showed that under quite general conditions a low-dimensional quasi-periodic flow will be unstable to a strange attractor. In a quasi-periodic flow, the equilibrium flow trajectory sits on an n-torus in the phase space, and the dynamics can be described with a finite number (n) of incommensurate temporal frequencies. Thus MWV is a particular example of a quasi-periodic flow. However, MWV is temporally periodic when viewed in the appropriate rotating frame; hence the transition to chaos studied by Brandstater & Swinney (1987) is directly from a limit cycle to chaos. The work of Ruelle & Takens does not apply in this case, nor is there any other known mathematical mechanism for such a transition.

We believe that a useful approach to the solution of this problem is to understand the nature of quasi-periodic instabilities of the rotating wave in general. This entails a closer look at the available experimental work, which reveals that the series of bifurcations in Taylor–Couette flow is more complicated than is suggested by the main sequence. At least two modulated waves of qualitatively different character have been observed in the laboratory (Coughlin *et al.* 1991), a fact which has not been reconciled with existing theory. In addition, there exist two conflicting nomenclatures for the MWV states which have been studied in detail, neither of which illuminates the physical nature of the modulated flow. Our numerical simulations of quasi-periodic Taylor–Couette flows, described in Part 2 (Coughlin & Marcus 1992), have shown that there is a simple, general framework in which all of these MWV states can be understood. In this paper, we lay down the mathematical basis for this understanding, based on the construction of the mathematical form of the solution. This work should be generally useful in any situation where rotating waves and quasi-periodic flows are found.

We use Floquet theory to deduce the functional form of instabilities of the rotating wave, and from this we derive the form of modulated wave solutions to the Navier–Stokes equations. The consequences of the mathematical structure of the flow, particularly the relation between the space–time symmetry as described by Rand and the form of the initial rotating wave and the Floquet mode, are discussed in detail. From this we are able to resolve the questions raised by current experimental and theoretical work. In the process, the parameters describing the solution are given a physical interpretation, with a detailed discussion deferred to Part 2.

To set our work in context, we present a brief review of the main sequence (discussion of the transition to chaos will be left to Part 2). A more complete review is given in DiPrima & Swinney, (1981), and a thorough discussion of the linear stability problem in Chandrasekhar (1961). The geometry of the apparatus is characterized by the radius ratio $\eta \equiv a/b$, where a and b are the inner and outer cylinder radii respectively, and the aspect ratio Γ which is equal to the height of the fluid column divided by the gap width. For the work discussed here, the gap is narrow with $\eta = 0.875$. We define a Reynolds number

$$R \equiv \Omega a(b-a)/\nu, \tag{1}$$

where Ω is the inner cylinder frequency, and ν is the kinematic viscosity. We use the usual cylindrical coordinates (r, θ, z), and non-dimensionalize by setting the gap width $(b-a)$, the inner cylinder velocity Ωa, and the density all equal to one.

At low R, circular Couette flow is a stable equilibrium, with the exact solution (for the infinite cylinder case) $v_r = v_z = 0$ and $v_\theta \equiv V(r)$, where

$$V(r) = A(r - b^2/r), \tag{2}$$

and $A = -\Omega\eta^2/(1-\eta^2)$. (In this paper we consider only the 'infinite cylinder' case, in which exact axial periodicity is imposed. For a justification of this assumption with regard to comparison with experiments see Part 2.) At $R \equiv R_c$ this flow becomes centrifugally unstable to Taylor vortex flow (TVF), which is steady, axisymmetric and periodic in the axial direction with wavelength λ. In our units, $R_c = 116.1$ for $\eta = 0875$. For R just a few per cent above R_c, a supercritical Hopf bifurcation leads to azimuthally travelling (rotating) waves on the Taylor vortices. We define the azimuthal wavenumber m_1 of wavy Taylor vortex flow (WVF) as the number of identical waves around the cylinder, and the fundamental frequency f_1 as the frequency with which successive wave peaks pass an observer in the lab frame. The phase speed $c_1 \equiv f_1/m_1$ is approximately independent of m_1 and R (King *et al.* 1982). For the gap widths considered here, c_1 is equal to roughly one third the inner cylinder speed.

At higher R, one observes modulated wavy vortex flow which is characterized by the presence of a second fundamental frequency incommensurate with f_1 (Gollub & Swinney 1975; Fenstermacher *et al.* 1979). Just as WVF is periodic in the inertial frame but steady in the frame rotating at speed c_1, MWV is quasi-periodic in the inertial frame and temporally periodic in the proper rotating frame. The modulation pattern may change the azimuthal symmetry of the flow, indicating that the new state must be described by two azimuthal wavenumbers as well as two temporal frequencies (Gorman & Swinney 1982; Gorman, Swinney & Rand 1981; Shaw *et al.* 1982).

In the experiments of Gorman & Swinney the modulated flow was identified with periodic flattening of the outflow boundaries between adjacent Taylor vortices. To distinguish this kind of MWV from other branches of quasi-periodic solutions we refer to it as 'GS flow'. For a given initial WVF state, a variety of azimuthal patterns were observed, all of which were found to be in agreement with the theoretical work of Rand (1982). Given an initial WVF with azimuthal wavenumber m_1, Rand showed that for MWV flows there exist an integer n and a basic period τ such that the flow in the frame rotating at the speed c_1 is invariant under the transformation $(\theta, t) \rightarrow (\theta + 2\pi n/m_1, t + \tau)$. Defining $s \geqslant 1$ as the fundamental azimuthal wavenumber of the modulated flow, the space–time symmetry of a given state is labelled by the three integers (m_1, n, s). Rand proved that for fixed m_1 and s there are a finite number (less than m_1) of modulated wavy flows with distinct labels (m_1, n, s). Shaw *et al.* (1982) conjectured that the GS flow was composed of two rotating waves, characterized by two azimuthal wavenumbers (defined as m_1 and m_2 respectively), and two wave speeds (c_1 and c_2). Inferring m_2 from the modulation pattern, they showed that the experimental power spectra were consistent with an approximately constant speed for each wave with $c_2/c_1 \approx \frac{4}{3}$. No explicit connection was made between the 'second wave' parameters (m_2, c_2) and the (n, s, τ) nomenclature of Rand, nor was the apparently unlimited range of m_2 reconciled with Rand's prediction of a finite number of observable symmetries.

A defining characteristic of GS flow is that there is always a peak $f_2 \equiv m_2 c_2$ in the power spectrum whose frequency shifts when the flow is observed in a rotating frame by an amount proportional to the frame speed. This property implies that the frequency f_2 is associated with a phase in θ. A modulated wave not having this property was observed by Zhang & Swinney (1985). The temporal power spectra showed a single strong peak associated with the modulation which remained fixed irrespective of the frame in which the flow was measured; hence, this flow was designated a 'non-travelling modulation'. We refer to this kind of modulated wave as a 'ZS flow'. No modulation signal was detectable at the inflow boundaries of the Taylor vortices, and there was no observed flattening at the outflow. This flow has not been set into the context of Rand's theory, which presumably must include it as a special case. The physical nature of the non-travelling mode, and why it is so, has not been explained.

A number of theoretical descriptions of modulated waves have been proposed in the literature. Swift *et al.* (1982) assumed the MWV to be a separable function of the time t and $\hat{\theta} \equiv \theta - c_1 t$, the azimuthal coordinate with respect to the rotating frame, and were able to correctly identify the locations of their observed spectral peaks. However, such a separable function is not an allowable solution of the Navier–Stokes equations, nor does it correctly predict the phase of the modulated wave. Ohji, Shionaya & Amagai (1986) proposed a functional form in which a single azimuthal mode is both amplitude and frequency modulated by a factor $\sin(k\hat{\theta} - w_M t)$ for some integer k, where ω_M is the observed frequency of modulation in the c_1-frame. While it is possible to represent the symmetries of the flow this way, it is unphysical because it includes only the fundamental azimuthal mode of the wave, and again functions of this form are not allowable solutions to the Navier–Stokes equations. Nor is it clear that one gains any physical insight by separating the modulation into amplitude and frequency components. In contrast, the functional form we shall derive in this paper is based on the structure of the linear eigenmodes of the rotating wave, and has a direct physical interpretation.

The rest of the paper is organized as follows: in §2 we use Floquet theory to derive the functional form of the solution, which is the same for all modulated rotating waves. The space–time symmetry properties, and the resolution of notational difficulties, follow directly. In §3, we show how the symmetry classes of Rand (1982) can be constructed from the functional form. Section 4 contains a discussion of how distinct branches of modulated wave solutions can be distinguished from each other by spectral data, augmented by general information about ZS and GS flows obtained from the numerical simulations. In §5 we show that a simple scaling relation allows a unique definition of the parameters of a particular state. Our conclusions are presented in §6. In the Appendix we describe the diagnostics used in our numerical computations of Taylor–Couette flow both to verify our hypotheses and to determine the parameters associated with the space–time structure of the flow. The code itself has been described by Marcus (1984*a*). In Part 2 we present detailed numerical results for both GS and ZS flows and describe the physical features which distinguish them.

2. Functional form

We derive the functional form of MWV beginning with the assumption that the rotating wave is linearly unstable to a Floquet mode. This assumption is used to clarify the derivation but our result does not depend on it. We shall ignore the radial and axial dependence of the solutions, as they are irrelevant to the discussion. We make reference to Taylor–Couette flow to illustrate the concepts presented. To simplify the notation, we denote an arbitrary scalar flow quantity in the rotating wave state as Q_W, and such a quantity in the modulated wave state as Q_M.

2.1. *Floquet theory*

We define a rotating wave as a solution to the Navier–Stokes equations with the symmetry

$$Q_{\mathrm{W}}(r,\theta+\delta\theta,z,t) = Q_{\mathrm{W}}(r,\theta,z,t-\delta\theta/c_1), \tag{3}$$

with c_1 the non-dimensionalized wave speed. For all time, the flow is also m_1-fold symmetric;

$$Q_{\mathrm{W}}(r,\theta+2\pi/m_1,z,t) = Q_{\mathrm{W}}(r,\theta,z,t), \tag{4}$$

for a positive integer $m_1 \geqslant 1$. The θ and t coordinates are coupled and periodic, so the solution must have the general form

$$Q_{\mathrm{W}}(r,\theta,z,t) = \sum_{j=-\infty}^{\infty} b_j(r,z)\exp[\mathrm{i}jm_1(\theta-c_1 t)]. \tag{5}$$

In WVF the azimuthal wavenumber m_1 is just the number of waves around the cylinder. In an experimental power spectrum there is a set of equally spaced peaks at multiples of $m_1 c_1$, thus we define the $f_1 \equiv m_1 c_1$ as the fundamental frequency of the rotating wave. For WVF, the phase speed c_1 is positive (the wave travels in the direction of the inner cylinder or mean azimuthal flow). Because Taylor vortices go unstable to WVF via Hopf bifurcation, the eigenmodes come in complex conjugate pairs, and the spectral sum in (5) has both positive and negative temporal frequencies $\pm jm_1 c_1$. The complex conjugate pairs are proportional to $\exp[\pm \mathrm{i}jm_1(\theta-c_1 t)]$; hence both frequency components describe waves travelling in the positive θ-direction. Waves which travel against the direction of the mean flow are not seen, neither in experiments nor in numerics. When the flow is observed in a frame rotating with angular speed c, the locations of spectral peaks are shifted to $jm_1(c_1-c)$. The flow will be time-independent if and only if $c = c_1$.

We define the new variable

$$\hat{\theta} = \theta - c_1 t, \tag{6}$$

such that $Q_{\mathrm{W}}(r,\hat{\theta},z)$ is steady. The Navier–Stokes equation linearized around Q_{W} has coefficients which are periodic in $\hat{\theta}$ and autonomous in t, thus we refer to the eigenmodes as Floquet modes. We assume that (i) there is no change in the axial structure of the solution, and (ii) the flow goes through a supercritical Hopf bifurcation to modulated waves (numerical data presented in Part 2 support this hypothesis for some quasi-periodic flows). Defining the critical Reynolds number for onset of modulation as R_{M}, the eigenmodes have the form

$$Q_{\mathrm{F}}(r,\hat{\theta},z,t) = \exp(\mathrm{i}m_2\hat{\theta})\exp(-\mathrm{i}\omega_{\mathrm{M}} t)\sum_{j=-\infty}^{\infty} \gamma_j(r,z)\exp(\mathrm{i}jm_1\hat{\theta}) + \text{c.c.}, \tag{7}$$

where ω_{M} is real at $R = R_{\mathrm{M}}$. In the c_1 frame the bifurcation is from a steady to a temporally periodic flow. Note that m_2 need not equal m_1; hence the azimuthal symmetry of the flow may be changed by the bifurcation. Because the solution is always periodic in θ there is an integer s, defined as the greatest common factor $\mathrm{GCF}(m_1, m_2)$, such that the flow is symmetric under $\theta \rightarrow \theta + 2\pi/s$.

2.2. *Nonlinear solution*

In order that the functional form of the fully nonlinear solution be invariant under the action of the Navier–Stokes equations, it must contain all the modes that might be generated as products of the eigenmode with itself or the rotating wave. Thus, MWV solutions can be written as a spectral sum of these products, and will have the form

$$Q_M(r, \hat{\theta}, z, t) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_{jk}(r, z) \exp[ijm_1 \hat{\theta}] \exp[ik(m_2 \hat{\theta} - \omega_M t)]. \tag{8}$$

Relative to the inertial frame, the flow is quasi-periodic, and we can use (8) to define two fundamental frequencies $f_1 \equiv m_1 c_1$ and $f_2 \equiv m_2 c_1 + \omega_M$. However, as is well known, the spectral data itself does not define uniquely the fundamental frequencies of a quasi-periodic signal, as any linearly independent combination (for example, $f_1 + f_2$ and $f_1 - f_2$) can be used as a basis for the entire set of peaks. In order to unambiguously specify the frequency of modulation, it must be identified with some physical flow characteristic. Here we define ω_M as the modulation frequency observed in the frame rotating at speed c_1, which we assume is known *a priori*.

A similar argument shows that the azimuthal wavenumbers m_1 and m_2 are not uniquely defined by the formal solution. Consider a spatial Fourier spectrum of the flow in (8) at some fixed time:

$$\tilde{Q}_M(l) = \left| \int_0^{2\pi} e^{-il\theta} Q_M \frac{d\theta}{2\pi} \right|^2. \tag{9}$$

This spectrum will have peaks at all values of the integer l such that

$$l = jm_1 + km_2 \tag{10}$$

for all integers j, k. All values of l will be multiples of s, the greatest common factor of m_1 and m_2. The set of pairs (m_1, m_2) which reproduces this set of peaks is not unique. The limitation on observable MWV flow patterns bifurcating from a given rotating wave follows from this fact. The functional form of the solution determines the modulation pattern observed in the laboratory; hence, for given m_1 and m_2, and $m_2' = m_2 + pm_1$ for integer p in (8) reproduces the same functional form and therefore the same space–time symmetry. If the modulation is a weak perturbation of the underlying rotating wave m_1 can be determined visually. However, to measure m_2, a way must be found of disentangling the 'background' m_1-fold symmetry from the total spatial signal (analogous to removing the background rotating wave frequency by transforming to the appropriate frame). While this has not yet been accomplished in the laboratory it is easy to do numerically, and results presented in Part 2 demonstrate the physical significance of m_2.

For now, we assume that c_1, ω_M, and the wavenumbers (m_1, m_2) are known. The Floquet mode in (7) consists of a spatial eigenfunction multiplied by a complex phase with the functional form of a rotating wave, and a phase speed relative to the inertial frame

$$c_2 \equiv c_1 + \omega_M / m_2. \tag{11}$$

The parameter c_2 turns out to be the same as the 'second wave speed' deduced by Shaw *et al.* (1982) in their study of Taylor–Couette flow, and m_2 is equal to their second wavenumber (modulo a multiple of m_1). To avoid confusion we do not refer to these parameters as descriptive of a second rotating wave, but rather as describing the Floquet mode associated with the modulation.

Using (11) to replace ω_M, and returning to the coordinates θ and t in the inertial frame, the flow in (8) becomes

$$Q_M(r, \theta, z, t) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_{jk}(r, z) \exp[ijm_1(\theta - c_1 t)] \exp[ikm_2(\theta - c_2 t)]. \tag{12}$$

Mathematically, the flow is a doubly periodic, non-separable function of the two phases $(\theta - c_1 t)$ and $(\theta - c_2 t)$. It is not correctly described as a pair of travelling waves, but there will be a doubly infinite set of peaks in the power spectra which have the properties of the spectra of travelling waves. There is a structural symmetry under $1 \leftrightarrow 2$, but because the matrix $a_{jk} \neq a_{kj}$ there is no physical equivalence.

The form of the quasi-periodic solution does not depend on the assumption of a supercritical Hopf bifurcation, as it is equivalent to a functional form based purely on the observed symmetries of the flow. We define α_1 and α_2 such that

$$m_1 \equiv \alpha_1 s, \tag{13}$$

$$m_2 \equiv \alpha_2 s; \tag{14}$$

hence α_1 and α_2 are relatively prime. Letting $l = j\alpha_1 + k\alpha_2$, we rewrite (8) as

$$Q_{\mathrm{M}}(r, \hat{\theta}, z, t) = \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} A_{lk}(r, z) \exp(\mathrm{i}ls\hat{\theta}) \exp(-\mathrm{i}k\omega_{\mathrm{M}} t), \tag{15}$$

where

$$A_{j\alpha_1+k\alpha_2,k} \equiv a_{j,k}. \tag{16}$$

This is the most general form for a flow which is temporally periodic when viewed in the c_1-frame, and spatially symmetric under $\theta \rightarrow \theta + 2\pi/s$. To have begun with this equation would however have obscured the real structure of the instability, as can be seen from the complicated relationship between the coefficients a_{jk} and A_{jk}. The matrix A_{jk} is full and has no symmetries, so there is no frame transformation which will simplify the space–time structure of the flow any further.

2.3. *Rotating frames*

The form of the solution in (12) can be used to prove that there is a doubly infinite family of rotating frame speeds $s_{n_1 n_2}$ in which the modulated wave appears to be temporally periodic. More specifically, there is an infinite family of possible periods T_J, and each T_J can be obtained in an infinite number of ways.

We define the period T_A as follows:

$$T_A = \frac{2\pi}{\alpha_1 \alpha_2 s |c_2 - c_1|}. \tag{17}$$

In any frame in which the flow is periodic, the periodicity must be a multiple of T_A; hence we define

$$T_J \equiv J T_A \tag{18}$$

for integer J.

Now define two integers n_1, n_2 such that for fixed J

$$J = n_1 \alpha_1 + n_2 \alpha_2. \tag{19}$$

The solutions to this equation define a one parameter family of integer pairs $\{n_1, n_2 = (J - n_1 \alpha_1)/\alpha_2\}$.

For fixed J and n_1, n_2 defined as above, in the frame rotating with speed

$$s_{n_1 n_2} = \frac{n_1 \alpha_1 c_1 + n_2 \alpha_2 c_2}{J}, \tag{20}$$

the flow has period T_J. In this frame, the solution takes the form

$$Q_{\mathrm{M}}(r, \theta_{n_1 n_2}, z, t) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} X_{jk}(r, z) \exp(\mathrm{i}js\theta_{n_1 n_2}) \exp(-\mathrm{i}k\omega_J t), \tag{21}$$

where $\theta_{n_1 n_2} \equiv \theta + s_{n_1 n_2} t$ and $\omega_J = 2\pi/T_J$. Thus, the basic functional form of quasi-periodic flow is preserved under this family of transformations, and $c_1 = s_{10}$ and $c_2 = s_{01}$ are not defined uniquely as frame speeds in which the flow looks periodic. Note that if m_2 is misidentified as some value $N = m_2 + pm_1$ for integer p, and c_2 is thus defined as $c_1 + \omega_M/N$, the flow will be periodic in this rotating frame, as can be proven by showing that $c_1 + \omega_M/N$ can be written in the form $s_{n_1 n_s}$ above.

3. Symmetry classes of modulated waves

In this section we use the functional form of the solution to re-derive Rand's results regarding the allowed symmetries of MWV. Equation (8) shows explicitly that, given a fixed value of m_1, all values of m_2 that differ by a multiple of m_1 will have the same symmetry. Equivalently, for fixed m_1, there are at most m_1 MWV states with distinct functional representations and therefore distinct modulation patterns. Rand (1982) proved this result, assuming that the modulation occurs as a Hopf bifurcation out of a rotating wave with known wavenumber m_1 and wave speed c_1. He defined the symmetry of MWV in terms of an integer n and a unique minimal period τ such that the flow is invariant under the transformation $(\theta, t) \rightarrow (\theta + 2\pi n/m_1, t+\tau)$. In experiments on Taylor–Couette flow, Gorman & Swinney (1982) identified τ as equal to s/m_1 times the modulation period observed in the frame comoving with the first wave.

We shall derive explicitly the relation between the spatial labels (m_1, n, s) and frequencies Ω_W and Ω_M used by Rand and the notation (m_1, c_1, m_2, c_2) used in this paper. These relations are summarized in table 1. For reference we include Rand's definitions

$$\Omega_M = 2\pi/\tau, \tag{22}$$

$$\Omega_W = s(c_1 + n\Omega_M/m_1). \tag{23}$$

We first consider the simple case where $m_1 = m_2 = s$ (the same relations hold if m_2 is any multiple of m_1). In this case, the flow must return to itself after a period τ with no azimuthal rotation; hence $n = 0$. The period τ is then equal to the period observed in the rotating frame; hence $\tau = 2\pi/\omega_M$. The fundamental frequencies are $\Omega_M = \omega_M$ and $\Omega_W = sc_1 = m_1 c_1$.

For $n \neq 0$, we substitute the phase shift $\theta \rightarrow \theta + \delta\theta$, $t \rightarrow t + \delta t$ into (8), from which it is clear that the shift in θ can be absorbed by a compensating shift in t if and only if $\delta\theta$ is a multiple of $2\pi/m_1$. The complex phase factor acquired in the shift must be equal to one; hence,

$$2\pi n \frac{m_2}{m_1} - \omega_M \tau = 2\pi n_r \tag{24}$$

for some integer n_r.

To simplify this expression we define

$$\alpha_2 = p\alpha_1 + \alpha_r \tag{25}$$

for integers p and α_r, with $p \geqslant 0$ and $0 \leqslant \alpha_r < \alpha_1$. This defines p and α_r uniquely. Replacing m_2/m_1 with α_r/α_1, and absorbing p into a redefinition of n_r, we find that $\omega_M \tau$ is equal to $2\pi/\alpha_1$ times an integer. Because τ is defined as a minimal period we set that integer equal to unity, hence

$$\tau \equiv 2\pi/\omega_M \alpha_1. \tag{26}$$

(This includes the special case $n = 0$, for which $\alpha_1 = 1$.)

Rand	This paper
m	$m_1 \equiv \alpha_1 s$
ω	c_1
s	$\mathrm{GCF}(m_1, m_2)$
	$m_2 \equiv \alpha_2 s$
n	$n\alpha_r - 1 = \alpha_1 \times \text{integer}$
	$\alpha_r \equiv \alpha_2 \bmod \alpha_1$
Ω_{M}	$\alpha_1 \omega_{\mathrm{M}}$
	$\omega_{\mathrm{M}} \equiv m_2(c_2 - c_1)$
Ω_{W}	$sc_1 + n\omega_{\mathrm{M}}$
τ	T_{M}/α_1
	$T_{\mathrm{M}} \equiv 2\pi/\omega_{\mathrm{M}}$

TABLE 1. Summary of relations between Rand's (1982) labels and those used here

To show that τ is unique, it is sufficient to show that it is in fact equal to the period T_A defined in (17), which is the frame-independent modulation period observed for any axisymmetric flow quantity. The axisymmetric modes in (8) satisfy

$$s(j\alpha_1 + k\alpha_2) = 0 \tag{27}$$

for an infinite set of integers j and k. Solutions of this equation take the form $k = K\alpha_1$, $j = -K\alpha_2$, where K is any integer. Thus, there will be peaks in a power spectrum of any axisymmetric flow quantity at the frequencies

$$k\omega_{\mathrm{M}} = K\alpha_1 \omega_{\mathrm{M}},$$

i.e. at multiples of the frequency $\alpha_1 \omega_{\mathrm{M}} \equiv \Omega_{\mathrm{M}}$. Given the definitions of τ and ω_{M}, we have

$$\Omega_{\mathrm{M}} = s\alpha_1 \alpha_2 (c_2 - c_1). \tag{28}$$

With τ defined as in (26), (24) reduces to

$$n\alpha_r - 1 = \alpha_1 n_r. \tag{29}$$

We remind the reader that n_r is arbitrary, and n must be chosen such that $0 \leqslant n < \alpha_1$. Note that $n \neq 0$ implies that $\alpha_1 \neq 1$, thus if we identify Rand's integer q with α_r, (29) proves his assertion that qn is not a multiple of $m_1/s \equiv \alpha_1$.

Clearly, the whole family $\{\alpha_2 = p\alpha_1 + \alpha_r; p \geqslant 0\}$ for fixed α_r will result in the same equation for n. This means that n and α_r contain equivalent information; hence if n is known, α_r can be deduced by a process of elimination. To derive all possible symmetry classes for a given m_1, we need only consider $1 \leqslant m_2 \leqslant m_1$ (in which case $\alpha_2 = \alpha_r$). The value of p cannot be determined from the symmetry, so (m_1, n, s) are not sufficient to determine m_2.

The equation for Ω_{W} implies that

$$m_1 c_1 = \alpha_1 \Omega_{\mathrm{W}} - n\Omega_{\mathrm{M}}. \tag{30}$$

Recall our definition $c_2 = c_1 + \omega_{\mathrm{M}}/m_2$. Substituting for c_1 we find that

$$m_2 c_2 = \alpha_2 \Omega_{\mathrm{W}} - (n\alpha_2 - 1)\frac{\Omega_{\mathrm{M}}}{\alpha_1}. \tag{31}$$

Equation (29) guarantees that the second term on the right-hand side will be equal to an integer times Ω_{M}.

The reverse relations can be written in a simpler way;

$$\Omega_W = sc_1 + n\omega_M, \tag{32}$$

and

$$\Omega_M = \alpha_1 \omega_M. \tag{33}$$

We emphasize that these symmetry classes do not distinguish among flows with the same value of m_2 (mod m_1). As we show in §5, the value of m_2 can be given a precise definition if properties of the flow other than the space–time symmetries are used.

4. Experiments

From the above analysis it is clear that the spatial symmetry of the modulated flow, determined by m_1 and m_2, constitutes a wavenumber selection problem which is undoubtedly dependent in a complicated way on the details of the flow. These quantities do not distinguish among distinct branches of modulated wave solutions, which may occur with any values of m_1 and m_2. We have found numerically that, for the Taylor–Couette system, distinct quasi-periodic solutions can be classified according to (i) the numerical value of c_2/c_1, which is approximately independent of m_1 and m_2, (ii) the characteristic amplitude of modulation, which may be crudely estimated as $|\Sigma_j a_{j0}|/|\Sigma_j a_{j1}|$ (cf. (12)), and (iii) the spatial structure of the Floquet eigenfunction and the modulation pattern that results from the interaction with the underlying WVF. In this section we explain our classification of the experimental data, deferring to §5 and the Appendix a discussion of how the numerical values of c_1, c_2, etc. are determined. Because the ratio c_2/c_1 is nearly constant for a given type of quasi-periodic flow, it can be used to help in the experimental identification of the value of m_2. In table 2 we present a summary of the experimental and numerical data reported here.

In the experiments of Zhang & Swinney (1985) a Taylor–Couette system with radius ratio 0.883 was used. The flow state was determined by flow visualization, and spectra were taken from the time series of scattered light intensity. With this technique, there is no precise quantitative relationship between the amplitude of a spectral peak and the strength of the corresponding mode (Savaş 1985). Zhang & Swinney observed a quasi-periodic flow in which the spectral peak associated with the modulation, which they called f_{s1}, was frame independent and thus not associated with an azimuthally travelling mode. Although in general the modulation frequency is not well defined by spectral peaks alone, in this case only one strong component (plus harmonics of the wave) was present in the spectrum, so the identification was valid. They determined that $m_1 = 5$ and that the flow was stable over a large range of parameter space; approximately $4R_c < R < 9R_c$ and $2.0 < \lambda < 3.5$, with some hysteresis at the stability boundaries.

For a numerical computation with $\lambda = 3$, $s = 5$ and $R = 7.5R_c$ we found a ZS modulated wave with $m_1 = m_2 = 5$ and a wave speed ratio $c_2/c_1 = 2.13$. These parameters are well inside the experimentally determined stability boundaries, thus we assume that this is the same flow seen by Zhang & Swinney. Because $m_2 = m_1$, the frequency ω_M is equal to the beat frequency $5(c_2 - c_1)$ between the two fundamentals, which does not shift when the flow is observed in a rotating frame. Numerically we have found that for ZS flows, owing to the extreme weakness of the modulation, the beat frequency dominates the power spectrum (after f_1 and its harmonics have been removed), which is consistent with Zhang & Swinney's observation that the largest

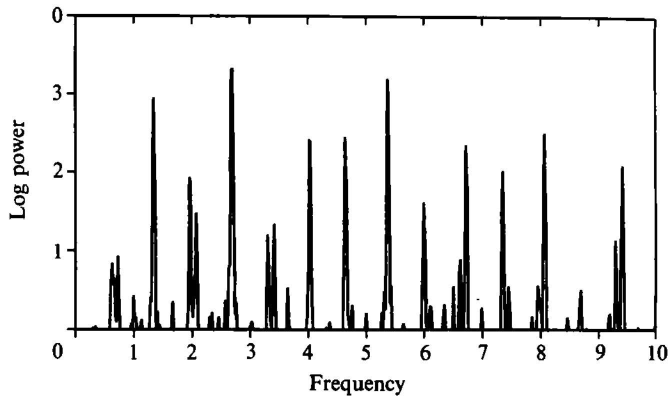

FIGURE 1. Experimental temporal power spectrum of a ZS flow at $\lambda = 2.5$, $R = 8.5R_c$, with $m_1 = 4$ and $m_2 = 12$, reproduced from Coughlin *et al.* 1991.

λ	m_1	m_2	R/R_c	c_1/Ω	T_M	c_2/c_1	Flow
3.0	5	5	7.5	0.353	22.1	2.13	ZS
2.5	4	12	8.5	0.338	10.6	2.02	ZS
2.5	4	4	9.8	0.331	107	1.31	GS
2.95	5	5	7.0	—	—	2.1	ZS
2.5	4	12	8.5	0.336	—	2.1	ZS
2.51	4	4	9.8	0.334	—	1.27	GS

TABLE 2. Data for numerical (upper three rows) and experimental (lower three rows) (Zhang & Swinney 1985, figure 2; Coughlin *et al.* 1991)

component in their spectrum, not identifiable as a harmonic of the WVF, was frame-independent. Thus

$$f_{s1} = s(c_2 - c_1), \tag{34}$$

which implies that

$$f_{s1}/f_1 = c_2/c_1 - 1 = 1.13. \tag{35}$$

From the graphical data in Zhang & Swinney we estimate a ratio of 1.1. The lack of a signal on the inflow boundaries is consistent with our numerical discovery that the quasi-periodic Floquet mode is localized in the vortex outflow region (see Part 2).

We have also computed a ZS flow at the parameter values $\lambda = 2.5, R = 8.5R_c$, with $m_1 = 4$ and $m_2 = 12$, which was subsequently observed in the laboratory (Coughlin *et al.* 1991). The experimental power spectrum is shown in figure 1. Although there are many peaks in the spectrum, the fundamental of the wave and its harmonics are prominent and can be distinguished easily. If we assume that, as with the $m_1 = m_2 = 5$ state, the largest peak not equal to a multiple of f_1 in this spectrum is the rotating-frame-independent beat frequency $m_2(c_2 - c_1)$, and that $c_2 \approx c_1$, then setting $m_2 = 12$ leads to a measured value of $c_2/c_1 = 2.15$, compared to the numerical value of 2.02. Independent confirmation of the modulation frequency ω_M, by taking data in the rotating frame, has not been done. In the laboratory the modulation appears as small ripples on the outflow boundary between adjacent Taylor vortices.

The numerically computed GS modes with $\eta = 0.875$ are characterized by phase speeds with $c_1/\Omega \approx \frac{1}{3}$, $c_2/c_1 \approx \frac{4}{3}$, and a weak dependence on the Reynolds number in agreement with the experiments of Shaw *et al.* Many harmonics of the modulation are found in the spectrum, but the problem of determining ω_M and m_2 is simplified by the fact that $m_2 (\mathrm{mod}\, m_1)$ can be observed directly from the spatial modulation pattern. Note that for these flows $f_2 \equiv m_2 c_2$ is not the most prominent peak associated with the modulation (see Gorman & Swinney 1979).

5. Scaling of spectral coefficients

While experimentally it may be difficult to determine the value of m_2, numerically we have found that it is directly expressed in the structure of the Floquet mode or, far beyond onset, in the spatial structure of the component of the flow associated with the modulation. (See figures 4 and 5 of Part 2). This implies that it must be possible to construct a mathematically unique definition of m_2 from the solution. Because redefinition of m_2 in (8) requires relabelling of the indices of a_{jk}, the physical significance of m_2 must be related to the structure of this coefficient matrix. In this section we show that plausible assumptions about the scaling of the spectral amplitudes a_{jk} can be used to identify m_2 from azimuthal power spectra of the flow.

We first note some observations about spatial Fourier spectra. Marcus (1984*a*, *b*) showed that the azimuthal spectrum of rotating waves is monotonic, with an approximate log–linear dependence on wavenumber (see figure 25 of that paper). (The numerically computed axial spectrum in Taylor–Couette flow, and also in curved channel flow (Finlay, Keller & Ferziger 1988), shows the same relation.) Consequently, if we scale the flow in (5) so that $b_0 \equiv 1$, then

$$|b_j| \sim \delta^{|j|}, \tag{36}$$

where $0 < \delta < 1$. (Note that $|b_j| = |b_{-j}|$.) We have found that the same holds for all the flows (rotating wave, modulated wave and chaotic) that we have computed. Near onset, this relation is simply understood in terms of amplitude expansions. The rotating wave disturbance has the form $\delta \exp[\mathrm{i}m_1(\theta - c_1 t)] f(r,z)$ for $\delta \ll 1$. Higher azimuthal modes are produced by nonlinear interaction of the fundamental with itself, so that modes proportional to $\exp[\mathrm{i}jm_1\theta]$ will scale as $\delta^{|j|}$. (The $j = 0$ mode is of order unity plus δ^2.) As there is no direct forcing of the flow at a particular azimuthal lengthscale, with increasing R the slope of the spectrum changes owing to the larger amount of energy in the high j modes, but the basic scaling relation remains as in (36).

Because θ and t are coupled in the rotating wave, the temporal and azimuthal power spectra are identical modulo a normalization factor. In the laboratory, assuming that peak strengths are directly correlated with the physical amplitudes, the largest spectral peak corresponds to f_1. For relatively weak modulation, as is observed in ZS and GS flows, f_1 and its lowest harmonics are still prominent in the MWV spectrum. Because c_1 is approximately constant, the measured value of f_1 provides a quantitative determination of m_1.

If MWV is measured in the frame rotating at speed c_1, determination of ω_{M} is trivial. From (11), $\omega_{\mathrm{M}} = f_2 - (m_2/m_1) f_1$. As is illustrated in figure 1, the large number of spectral peaks makes determination of the correct value f_2, and therefore of m_2, difficult without the use of additional information (even with c_2 known from the nature of the flow). Thus spectral data alone do not provide enough information to deduce the value of m_2. Numerically, we take advantage of the spatial information available to separate directly the piece of the flow that is quasi-periodic from the piece that is steady in the c_1-frame, as follows.

In the frame rotating at speed c_1 the flow is periodic, so we define the mean as the steady-state component which is obtained by time-averaging the flow over one modulation period. We defined the quasi-periodic disturbance as the total flow minus the steady-state in the same frame. This definition is specific to the c_1-frame and is chosen to ensure that, to first order in the modulation amplitude, the steady-state flow is equal to the unstable wave and the quasi-periodic disturbance is equal to the Floquet mode. This would not be true if the time-averaging were done in any other rotating frame in which the MWV is periodic.

Formally, a mean flow quantity $\bar{Q}_M$ is defined as

$$\bar{Q}_M(r,\hat{\theta},z) \equiv \frac{1}{T_M}\int_0^{T_M} Q_M(r,\hat{\theta},z,t)\,\mathrm{d}t, \tag{37}$$

which becomes, in the language of (8),

$$\bar{Q}_M(r,\hat{\theta},z) \equiv \sum_{j=-\infty}^{\infty} a_{j0}(r,z)\exp[\mathrm{i}jm_1\hat{\theta}]. \tag{38}$$

For the total velocity field $\boldsymbol{v}$, the mean $\bar{\boldsymbol{v}}$ is thus

$$\bar{\boldsymbol{v}}(r,\tilde{\theta},z) \equiv \frac{1}{T_M}\int_0^{T_M} \boldsymbol{v}(r,\hat{\theta},z,t)\,\mathrm{d}t, \tag{39}$$

and the quasi-periodic piece is

$$\boldsymbol{v}_{qp}(r,\hat{\theta},z,t) \equiv \boldsymbol{v}(r,\hat{\theta},z,t)-\bar{\boldsymbol{v}}(r,\hat{\theta},z). \tag{40}$$

Note that $\bar{\boldsymbol{v}}$ always has the symmetry of the underlying rotating wave. The spatial periodicity of $\boldsymbol{v}_{qp}$ is determined by s (which for the numerical solutions presented here is identical to m_1). It is not symmetric under $\theta \rightarrow \theta + 2\pi/m_2$, even at onset, unless of course $m_2 = m_1$.

At onset $\boldsymbol{v}_{qp}$ is equal to the Floquet mode which is of the form given in (7). Our numerical results indicate that, consistent with the observation that spatial Fourier spectra of Taylor–Couette flows generically show a log–linear scaling, the spectral coefficients γ_j in the Floquet eigenfunction scale as

$$|\gamma_j| \sim \rho^{|j|}, \tag{41}$$

for some $0 < \rho < 1$. This means that, for Q_F itself, the largest peaks in the azimuthal Fourier spectrum are γ_0 and γ_0^*, which multiply the modes $\exp(\pm \mathrm{i}m_2\theta)$. Equivalently, the azimuthal power spectrum of the quasi-periodic velocity at fixed time should peak at the azimuthal mode equal to m_2, as is shown in Appendix A §A.2 and confirmed by numerical data for two ZS flows. Graphically, this means that in the frame rotating with speed c_1 the Floquet eigenfunction consists of a time-independent function with wavelength $2\pi/m_1$ spatially modulated by a single Fourier mode proportional to $\exp\{\mathrm{i}m_1[\hat{\theta}-(c_2-c_1)t]\}$. This modulating function 'travels' in the θ-direction with angular velocity c_2-c_1. For Taylor–Couette flow (as illustrated in Part 2) plots of any component of $\boldsymbol{v}_{qp}$ demonstrate that the value of m_2 so defined is not at all ambiguous.

Monotonic decay of the spectral amplitudes can also be used in the laboratory to verify that an estimated value of c_2 is really the phase speed associated with the Floquet mode. In the frame rotating at speed c_2, a time series taken at a fixed point (r,θ,z) will have the form

$$f(t) = \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} a_{jk}(r,z)\exp[-\mathrm{i}jm_1(c_1-c_2)t]\exp[\mathrm{i}km_2\theta]\exp[\mathrm{i}jm_1\theta]. \tag{42}$$

Setting $\zeta_j = \exp[ijm_1\theta](\sum_k a_{jk}(r,z)\exp[ikm_2\theta])$, we find

$$f(t) = \sum_{j=-\infty}^{\infty} \zeta_j \exp[-ijm_1(c_1-c_2)t], \tag{43}$$

hence the temporal spectrum in this frame scales with the index j, and will therefore be monotonic. In any other frame in which the flow looks periodic the temporal spectra will not have this monotonic decay for all j.

6. Conclusion

We have shown that the explicit functional form of quasi-periodic flow can be determined from simple considerations of the mathematical structure of linear eigenmodes of rotating waves in Taylor–Couette flow, a procedure which can be easily generalized to other systems having a similar class of solutions. Using an explicit representation for the MWV flow, we have been able to clear up a number of inconsistencies in notation and interpretation of experiments in existing work. We have also shown that the known symmetry classes of quasi-periodic flow, as discussed by Rand (1982), can be derived directly from the solution, and that physically distinct flows can have the same symmetry. With regard to the latter, there are two points to make: (i) there are distinct branches of MWV which are distinguished by the numerical values of the phase speed of the Floquet mode and the amplitude of modulation, not by symmetry; and (ii) for a given class of MWV, there can be flows which have the same value of m_1 but different values of m_2, which therefore have identical symmetry but are not physically equivalent. We have shown that a unique, well-defined, and physically meaningful value of m_2 can be determined from the scaling of amplitudes in the spectral representation of the flow, with numerical confirmation of the hypotheses used in the analysis. We believe that a clearer understanding of the mathematical representation of quasi-periodic solutions to the Navier–Stokes equations, and of the fact that there exist several branches of such solutions, will be of further use in both experimental work and bifurcation theory.

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Appendix. Numerical diagnostics

In this Appendix we discuss the numerical diagnostics used to verify the statements made in this paper about the functional form of MWV flows in the Taylor–Couette system. We have used a pseudospectral, initial-value code to solve the Navier–Stokes equations for incompressible flow in a cylindrical geometry (Marcus 1984*a*, *b*; Coughlin & Marcus 1991). The flow is constrained to be periodic in the axial direction with wavelength λ, and in the azimuthal direction with wavelength $2\pi/s$. To describe the temporal properties of the flow we use both the phase speed c_2 and the modulation frequency ω_M, the latter being directly comparable to experimentally measured values. The modulation period $T_M \equiv 2\pi/\omega_M$ is the periodicity observed in the c_1-frame, measured in our dimensionless units. We will use the notation (m_1, m_2) to denote a modulated wave with these azimuthal wavenumbers.

A.1. *Analysis of time series*

We represent the solution using Fourier modes in the periodic directions, and Chebyshev polynomials for the radial coordinate. For all the computations we define $m_1 \equiv s$, which reduces the number of azimuthal Fourier modes needed to represent the wavy vortex solution. This restricts the numerical modulated wave solutions to the case $m_2/m_1 = j$ where j is an integer. (We have observed flows with both $j = 1$ and $j \neq 1$.) Throughout this Appendix, we will ignore the discretization in time, and assume that time series are taken at some fixed radial point.

We define all variables with respect to the inertial frame. An arbitrary scalar flow quantity is represented as

$$Q(r,\theta,z,t) = \sum_{n=-N+1}^{N-1} \sum_{j=-J+1}^{J-1} f_{nj}(r,t) \exp[2\pi i n z/\lambda] \exp[ijs\theta]. \tag{A 1}$$

For all our computations, the flow is adequately resolved with $N \leqslant 32$, $J \leqslant 32$, and 33 radial modes

Knowing the functional form of the solutions allows us to predict the temporal properties of both rotating waves and modulated waves. Time series of the coefficients of spatial Fourier modes allow us both to verify the deductions and to determine the time constants. For our time series, we use the modes ($n = 1$; $j = 0, 1, 2, 3, 4$) and ($j = 1$; $n = 2, 3, 4$) at a fixed radial point, as well as the torque at the outer cylinder, total energy, and the total angular momentum.

For WVF, the modes b_j of (5) and the spatial Fourier modes are related by

$$b_j(r,z) \exp[-ijsc_1 t] = \sum_n f_{nj}(r,t) \exp[2\pi i n z/\lambda], \tag{A 2}$$

which implies that

$$f_{nj}(r,t) = c_{nj}(r) \exp[-ijsc_1 t], \tag{A 3}$$

where c_{nj} is a complex function of r.

The signature of a converged rotating wave is that any axisymmetric quantity (such as the energy) will be steady state, while any non-axisymmetric flow variable will be periodic with period $2\pi/sc_1$. To determine the speed of the waves, we use the function

$$h_{nj}(r,t,\hat{t}) = \frac{1}{\hat{t}} \log\left(\frac{f_{nj}(r,t+\hat{t})}{f_{nj}(r,t)}\right). \tag{A 4}$$

For rotating waves this expression reduces to

$$h_{nj}(r,t,\hat{t}) = -ijsc_1, \tag{A 5}$$

which is independent of r, n, t and $\hat{t}$, and linear in j. As the flow converges from the initial condition to WVF, ih_{nj}/js converges as expected to a constant, which is equal to c_1. (Any dependence of h_{nj} on n would indicate the presence of waves travelling in the axial direction.)

Besides being a parameter of physical interest, the wave speed c_1 is a useful numerical diagnostic for monitoring the convergence of the solution. We continue the computation until c_1 is constant to at least one part in 10^4. The dependence of c_1 on the timestep δt is consistent with our second-order accurate algorithm (Marcus 1984*a*). A change in the spatial resolution produces a change in the wave speed of less than $O(10^{-5})$, which is on the order of truncation error.

The numerical signature of the onset of modulated waves is periodic oscillation in any axisymmetric quantity. For flows with $m_1 \equiv s$, m_2/m_1 = integer, the modulation frequency with respect to the c_1-frame, ω_M, is equal to the modulation frequency of axisymmetric flow quantities. Thus the period of the energy etc., which is easily determined from time series, gives us $\omega_M \equiv 2\pi/T_M$ directly.

The time dependence of the quasi-periodic Fourier modes is most readily deduced from the representation in (15). The coefficients A_{jk} are related to the f_{nj} by

$$\sum_n f_{nj}(r,t) \exp[2\pi i n z/\lambda] = \exp[-ijsc_1 t] \sum_k A_{jk}(r,z) \exp[-ik\omega_M t]. \tag{A 6}$$

The time series are therefore of the form

$$f_{nj}(r,t) = \exp[-ijsc_1 t] \sum_k c_{njk}(r) \exp[-ik\omega_M t], \tag{A 7}$$

where the c_{njk} again are complex functions of r.

The dependence on c_1 can be removed from these time series by taking the modulus of f_{nj}. For converged MWV we measure the period T_M by locating the extrema of $|f_{nj}|$ for the set of Fourier modes specified above. The accuracy in T_M is always plus or minus one timestep δt, which is of order $\delta t/T_M \sim 10^{-3}$. As the flow converges to a truly quasi-periodic state, the extrema of the time series converge to constant values. Transients in this system are generally on the order of a few modulation periods, except very close to a bifurcation.

Given ω_M, the wave speed c_1 can be calculated from (A 4) by setting $\hat{t} = T_M$. As with WVF, convergence to quasi-periodic flow can be monitored by verifying that h_{nj} goes to a constant value independent of t. Note that this procedure does not completely specify c_1, as the right branch of the logarithm in (A 4) must be chosen. In practice, we guess an approximate value for c_1 from continuity with the WVF, and use (A 4) to refine the estimation.

We have run a number of computations of (stable) MWV for several tens of modulation periods, and find that there is no drift in either the modulation period or the wave speed. On the contrary, quantities such as the amplitudes of the extrema of $|f_{nj}|$ continue to converge to constant values, implying that the time integration remains accurate. Equivalently, the function $h_{nj}(r,t,\hat{t})$ is independent of t to within a monotonically decreasing error. In the maxima and minima of time series for any flow quantity, there is either a monotonic convergence to the equilibrium values, or oscillatory convergence within a monotonically decreasing envelope.

A.2. *Numerical estimates of the scaling of A_{jk}*

In this section we discuss our numerical corroboration of the scaling derived in §5. In general, numerical time series are too short to produce power spectra with accurate peak locations and amplitudes. However, the time series themselves are very accurate, and because the time dependence of all Fourier modes is known explicitly, they can be used to isolate any particular amplitude A_{jk}. These coefficients are functions of r and z, thus we write

$$A_{jk}(r,z) = \sum_n c_{njk}(r) \exp[in2\pi z/\lambda]. \tag{A 8}$$

Note that, if a flow has shift-and-reflect symmetry, $|c_{n,-j,-k}| = |c_{njk}|$.

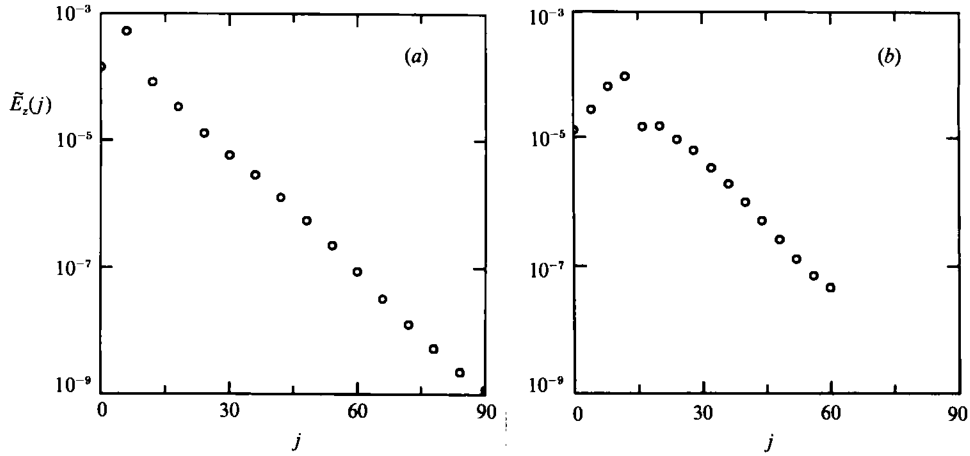

FIGURE 2. Azimuthal Fourier spectrum of the axial component of the quasi-periodic velocity (at fixed time)

$$\tilde{E}_z(j) = \frac{1}{4\pi\lambda}\int_a^b\int_0^\lambda\left|\int_0^{2\pi} e^{-ij\theta}(\hat{e}_z\cdot v_{qp})\,d\theta\right|^2 r\,dr\,dz,$$

for ZS flows with (*a*) $m_1 = m_2 = 6$, and (*b*) $m_1 = 4$, $m_2 = 12$.

To eliminate the dependency on the axial mode and radial point, we construct the azimuthal spectrum of the Floquet mode itself, or its nonlinear analogue v_{qp}. We define

$$\beta_{jk} \equiv \int_0^\lambda\int_a^b |A_{jk}(r,z)|^2\, r\,dr, \tag{A 9}$$

or equivalently,

$$\beta_{jk} \equiv \sum_n \int_a^b |c_{njk}(r)|^2\, r\,dr. \tag{A 10}$$

The azimuthal Fourier spectrum of the quasi-periodic mode at fixed t is then

$$\beta_j(t) \equiv \sum_{k\neq 0} \beta_{jk}\,\overline{\exp}\,[-ik\omega_M t]. \tag{A 11}$$

According to our hypothesis, the structure of the Floquet mode is $\exp[is\alpha_2\theta - \omega_M t]$ times a function with a monotonically decaying azimuthal spectrum, so the spectrum β_j should peak at $j = \alpha_2$. In figure 2 we present numerically computed spectra of $\hat{e}_z\cdot v_{qp}$ for the cases $\alpha_2 = 1$ and $\alpha_2 = 3$, verifying this prediction.

Errors in the evaluation of the quasi-periodic mode itself can be estimated by computing the spatial Fourier spectra of the residual

$$|v(r,\theta,z,t=0) - v(r,\theta + c_1 T_m, z, t = T_m)|^2.$$

For perfectly converged MWV solutions, this quantity is equal to zero. An inaccurate value of c_1, an inaccurate value of T_M, or a lack of convergence to the equilibrium can cause finite errors. The latter effect is always dominant. The ratio of this residual to the spatial Fourier spectrum of the quasi-periodic mode for fixed n, j serves as an estimate of signal-to-noise in the flow. For all the work presented here and in Part 2 this ratio is on the order of 10^{-2} or less.

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